SIMULTANEOUS UNITARY EQUIVALENCE TO CARLEMAN OPERATORS WITH ARBITRARILY SMOOTH KERNELS

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ABSTRACT. In this paper, we describe families of those bounded linear operators on a separable Hilbert space that are simultaneously unitarily equivalent to integral operators on $L_2(\mathbb{R})$ with bounded and *arbitrarily* smooth Carleman kernels. The main result is a qualitative sharpening of an earlier result of [7].

1. INTRODUCTION. MAIN RESULT

Throughout, $\mathcal{H}$ will denote a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$, $\mathfrak{R}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathbb{C}$, and $\mathbb{N}$, and $\mathbb{Z}$, the complex plane, the set of all positive integers, the set of all integers, respectively. For an operator A in $\mathfrak{R}(\mathcal{H})$, A^* will denote the Hilbert space adjoint of A in $\mathfrak{R}(\mathcal{H})$.

Throughout, $C(X,B)$, where B is a Banach space (with norm $\|\cdot\|_B$), denote the Banach space (with the norm $\|f\|_{C(X,B)} = \sup\limits_{x\in X}\|f(x)\|_B$) of continuous B-valued functions defined on a locally compact space X and *vanishing at infinity* (that is, given any $f\in C(X,B)$ and $\varepsilon>0$, there exists a compact subset $X(\varepsilon,f)\subset X$ such that $\|f(x)\|_B<\varepsilon$ whenever $x\notin X(\varepsilon,f)$).

Let $\mathbb{R}$ be the real line $(-\infty,+\infty)$ with the Lebesgue measure, and let $L_2 = L_2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable complex-valued functions on $\mathbb{R}$ equipped with the inner product

$$\langle f,g\rangle = \int_{\mathbb{R}} f(s)\overline{g(s)}\,ds$$

and the norm $\|f\| = \langle f,f\rangle^{\frac{1}{2}}$.

A linear operator $T: L_2\to L_2$ is said to be *integral* if there exists a measurable function $\boldsymbol{T}$ on the Cartesian product $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$, a *kernel*, such that, for every $f\in L_2$,

$$(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{T}(s,t)f(t)\,dt$$

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for almost every s in $\mathbb{R}$. A kernel $\boldsymbol{T}$ on $\mathbb{R}^2$ is said to be *Carleman* if $\boldsymbol{T}(s,\cdot) \in L_2$ for almost every fixed s in $\mathbb{R}$. An integral operator with a kernel $\boldsymbol{T}$ is called *Carleman* if $\boldsymbol{T}$ is a Carleman kernel. Every Carleman kernel, $\boldsymbol{T}$, induces a *Carleman function* $\boldsymbol{t}$ from $\mathbb{R}$ to L_2 by $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}(s,\cdot) \in L_2$.

The integral representability problem for linear operators stems from the work [10] of von Neumann, and is now well enough understood. The problem involves the question: which operators are unitarily equivalent to an integral operator? Now we recall a characterization of Carleman representable operators to within unitary equivalence [5, p. 99], [3, Section 15]:

Proposition 1. *A necessary and sufficient condition that an operator $S \in \mathfrak{R}(\mathcal{H})$ be unitarily equivalent to an integral operator with Carleman kernel is that there exist an orthonormal sequence $\{e_n\}$ such that*

$$\|S^* e_n\|_{\mathcal{H}} \to 0 \quad \textit{as } n \to \infty$$

(or, equivalently, that 0 belong to the right essential spectrum of S).

Given any non-negative integer m, we impose on a Carleman kernel $\boldsymbol{K}$ the following smoothness conditions:

(i) the function $\boldsymbol{K}$ and all its partial derivatives on $\mathbb{R}^2$ up to order m are in $C(\mathbb{R}^2, \mathbb{C})$,

(ii) the Carleman function $\boldsymbol{k}$, $\boldsymbol{k}(s) = \overline{\boldsymbol{K}(s,\cdot)}$, and all its (strong) derivatives on $\mathbb{R}$ up to order m are in $C(\mathbb{R}, L_2)$.

Definition 1. A function $\boldsymbol{K}$ that satisfies Conditions (i), (ii) is called a SK^m*-kernel* [7].

Now we are in a position to formulate our result on simultaneous integral representability of operator families by SK^m-kernels.

Proposition 2 ([7])**.** *If for a countable family $\{B_r \mid r \in \mathbb{N}\} \subset \mathfrak{R}(\mathcal{H})$ there exists an orthonormal sequence $\{e_n\}$ such that*

$$\sup_{r\in\mathbb{N}} \|B_r^* e_n\|_{\mathcal{H}} \to 0 \quad \textit{as } n \to \infty,$$

then for each fixed non-negative integer m there exists a unitary operator $U_m : \mathcal{H} \to L_2$ such that all the operators $U_m B_r U_m^{-1}$ $(r \in \mathbb{N})$ are bounded Carleman operators having SK^m-kernels.

In [7], there is a counterexample which shows that Proposition 2 may fail to be true if the family $\{B_r\}$ is not countable.

The purpose of this paper is to restrict the conclusion of Proposition 2 to *arbitrarily* smooth Carleman kernels. Now define these kernels.

Definition 2. We say that a function $\boldsymbol{K}$ is a SK^∞*-kernel* ([8], [9]) if it is a SK^m-kernel for each non-negative integer m.

Theorem. *If for a countable family* $\{B_r \mid r \in \mathbb{N}\} \subset \mathfrak{R}(\mathcal{H})$ *there exists an orthonormal sequence* $\{v_n\}$ *such that*

$$\sup_{r\in\mathbb{N}} \|B_r^* v_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty, \tag{1}$$

then there exists a unitary operator $U_\infty : \mathcal{H} \to L_2$ *such that all the operators* $U_\infty B_r U_\infty^{-1}$ $(r \in \mathbb{N})$ *are Carleman operators having* SK^∞*-kernels.*

This theorem, which is our main result, will be proved in the next section of the present paper. The proof yields an explicit construction of the unitary operator $U_\infty : \mathcal{H} \to L_2$. The construction of U_∞ is independent of those spectral points of B_r ($r \in \mathbb{N}$) that are different from 0, and is defined by $U_\infty f_n = u_n$ ($n \in \mathbb{N}$), where $\{f_n\}$, $\{u_n\}$ are orthonormal bases in $\mathcal{H}$ and L_2, respectively, whose elements can be explicitly described in terms of the operator family.

2. Proof of Theorem

The proof has two steps.

Step 1. Assume that

$$\sup_{r\in\mathbb{N}} \|B_r\| \le 1.$$

This is a harmless assumption, involving no loss of generality; just replace B_r with $\|B_r\| > 1$ by $\dfrac{B_r}{\|B_r\|}$. Find a subsequence $\{e_k\}_{k=1}^\infty$ of the sequence $\{v_n\}$ in (1) so that

$$\begin{aligned} \sum_k \sup_{r\in\mathbb{N}} \|S_r^* e_k\|_{\mathcal{H}}^{\frac14} &\le \sum_k \sup_{r\in\mathbb{N}} \|r S_r^* e_k\|_{\mathcal{H}}^{\frac14} \\ &= \sum_k \sup_{r\in\mathbb{N}} \|B_r^* e_k\|_{\mathcal{H}}^{\frac14} = M < \infty, \end{aligned} \tag{2}$$

where $S_r = \dfrac{1}{r} B_r$ ($r \in \mathbb{N}$) (the sum notation $\sum\limits_k$ will always be used instead of the more detailed symbol $\sum\limits_{k=1}^\infty$). For each r, let

$$Q_r = (1-E)S_r, \quad J_r = S_r^* E, \tag{3}$$

where E is the orthogonal projection onto the closed linear span H of the e_k's, and observe that

$$S_r = Q_r + J_r^*. \tag{4}$$

Assume, with no loss of generality, that $\dim(1-E)H = \infty$, and let $\{e_k^\perp\}_{k=1}^\infty$ be any orthonormal basis for $(1-E)H$. Let $\{f_n\}_{n=1}^\infty$ denote any basis in $\mathcal{H}$ consisting of the elements of the set $\{e_k\} \cup \left\{e_k^\perp\right\}$. It follows from (2) that

$$\sum_n \|J_r f_n\|_{\mathcal{H}} = \sum_k \|J_r e_k\|_{\mathcal{H}} \le \sum_k \sup_{r\in\mathbb{N}} \|S_r^* e_k\|_{\mathcal{H}} \le M^4,$$

and hence that J_r and J_r^* are Hilbert–Schmidt operators, for each r.

For each $h \in \mathcal{H}$, let

$$d(h) = \sup_{r\in\mathbb{N}} \|J_r h\|_{\mathcal{H}}^{\frac{1}{4}} + \sup_{r\in\mathbb{N}} \|J_r^* h\|_{\mathcal{H}}^{\frac{1}{4}} + \sup_{r\in\mathbb{N}} \|\Gamma_r h\|_{\mathcal{H}}, \tag{5}$$

where, for each r,

$$\Gamma_r = \Lambda S_r, \text{ and } \Lambda = \sum_k \frac{1}{k} \left\langle \cdot, e_k^{\perp} \right\rangle_{\mathcal{H}} e_k^{\perp}. \tag{6}$$

It is clear that Λ and Γ_r ($r \in \mathbb{N}$) are Hilbert-Schmidt operators on $\mathcal{H}$. Prove that

$$d(e_k) \to 0 \quad \text{as } k \to \infty. \tag{7}$$

Using known facts about Hilbert–Schmidt operators (see [2, Chapter III]), write the following relations

$$\begin{aligned}\sum_r \sup_{r\in\mathbb{N}} \|J_r^* e_k\|_{\mathcal{H}}^2 &\le \sum_r \sum_k \|J_r^* e_k\|_{\mathcal{H}}^2 \le \sum_r |J_r^*|_2^2 = \sum_r |J_r|_2^2 \\ &= \sum_r \sum_k \|J_r e_k\|_{\mathcal{H}}^2 = \sum_r \sum_k \|S_r^* e_k\|_{\mathcal{H}}^2 \\ &\le \sum_r \frac{1}{r^2} \sum_k \sup_{r\in\mathbb{N}} \|r S_r^* e_k\|_{\mathcal{H}}^2 \le \frac{M^8 \pi^2}{6},\end{aligned} \tag{8}$$

where $|\cdot|_2$ is the Hilbert–Schmidt norm. Observe also that

$$\begin{aligned}\sum_k \sup_{r\in\mathbb{N}} \|\Gamma_r e_k\|_{\mathcal{H}}^2 &\le \sum_r \sum_k \|\Gamma_r e_k\|_{\mathcal{H}}^2 \\ \le \sum_r |\Gamma_r|_2^2 &= \sum_r |\Gamma_r^*|_2^2 = \sum_r \sum_n \|S_r^* \Lambda f_n\|_{\mathcal{H}}^2 \\ \le \sum_r \frac{1}{r^2} \sum_k \left\|\Lambda e_k^{\perp}\right\|_{\mathcal{H}}^2 &= \sum_r \frac{1}{r^2} \sum_k \frac{1}{k^2} = \frac{\pi^4}{36}.\end{aligned} \tag{9}$$

Then (7) follows immediately from (8), (9), (2), and (3).

Notation. If an equivalence class $f \in L_2$ contains a function belonging to $C(\mathbb{R}, \mathbb{C})$, then we shall use $[f]$ to denote that function.

Take any orthonormal basis $\{u_n\}$ for L_2 which satisfies conditions:

(a) the terms of the derivative sequence $\left\{[u_n]^{(i)}\right\}$ are in $C(\mathbb{R}, \mathbb{C})$, for each i (here and throughout, the letter i is reserved for all non-negative integers),

(b) $\{u_n\} = \{g_k\}_{k=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$, where $\{g_k\}_{k=1}^{\infty} \cap \{h_k\}_{k=1}^{\infty} = \varnothing$, and, for each i,

$$\sum_k H_{k,i} < \infty \quad \text{with } H_{k,i} = \left\|[h_k]^{(i)}\right\|_{C(\mathbb{R},\mathbb{C})} \quad (k \in \mathbb{N}), \tag{10}$$

(c) there exist a subsequence $\{x_k\}_{k=1}^{\infty} \subset \{e_k\}$ and a strictly increasing sequence $\{n(k)\}_{k=1}^{\infty}$ of positive integers such that, for each i,

$$\sum_k d(x_k)\,(G_{k,i}+1) < \infty \quad \text{with } G_{k,i} = \left\| [g_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \quad (k \in \mathbb{N}), \tag{11}$$

$$\sum_k k H_{n(k),i} < \infty. \tag{12}$$

Remark. Let $\{u_n\}$ be an orthonormal basis for L_2 such that, for each i,

$$[u_n]^{(i)} \in C(\mathbb{R},\mathbb{C}) \quad (n \in \mathbb{N}), \tag{13}$$

$$\left\| [u_n]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le D_n A_i \quad (n \in \mathbb{N}), \tag{14}$$

$$\sum_k D_{n_k} < \infty, \tag{15}$$

where $\{D_n\}_{n=1}^{\infty}$, $\{A_i\}_{i=0}^{\infty}$ are sequences of positive numbers, and $\{n_k\}_{k=1}^{\infty}$ is a subsequence of $\mathbb{N}$ such that $\mathbb{N} \setminus \{n_k\}_{k=1}^{\infty}$ is a countable set. By (7), the basis $\{u_n\}$ satisfies Conditions (a)-(c) with $h_k = u_{n_k}$ ($k \in \mathbb{N}$) and $\{g_k\}_{k=1}^{\infty} = \{u_n\} \setminus \{h_k\}_{k=1}^{\infty}$.

To show the existence of a basis $\{u_n\}$ satisfying (13)-(15), consider a Lemarié-Meyer wavelet,

$$u(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(\frac{1}{2}+s)} \operatorname{sign}\xi\, b(|\xi|)\, d\xi \quad (s \in \mathbb{R}),$$

with the bell function b belonging to $C^{\infty}(\mathbb{R})$ (for construction of the Lemarié-Meyer wavelets we refer to [6], [1, § 4], [4, Example D, p. 62]). In this case, u belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, and hence all the derivatives $[u]^{(i)}$ are in $C(\mathbb{R},\mathbb{C})$). The "mother function" u generates an orthonormal basis for L_2 by

$$u_{jk}(s) = 2^{\frac{j}{2}} u(2^j s - k) \quad (j,\ k \in \mathbb{Z}).$$

Rearrange, in a completely arbitrary manner, the orthonormal set $\{u_{jk}\}_{j,\,k\in\mathbb{Z}}$ into a simple sequence, so that it becomes $\{u_n\}_{n\in\mathbb{N}}$. Since, in view of this rearrangement, to each $n \in \mathbb{N}$ there corresponds a unique pair of integers j_n, k_n, and conversely, we can write, for each i,

$$\left\| [u_n]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} = \left\| [u_{j_n k_n}]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le D_n A_i,$$

where

$$D_n = \begin{cases} 2^{j_n^2} & \text{if } j_n > 0, \\ \left(\dfrac{1}{\sqrt{2}} \right)^{|j_n|} & \text{if } j_n \le 0, \end{cases} \qquad A_i = 2^{\left(i+\frac{1}{2}\right)^2} \left\| [u]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})}.$$

Whence it follows that if $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ is a subsequence such that $j_{n_k} \to -\infty$ as $k \to \infty$, then

$$\sum_k D_{n_k} < \infty.$$

Thus, the basis $\{u_n\}$ satisfies Conditions (13)-(15).

Let us return to the proof. Let $\left\{x_k^{\perp}\right\}_{k=1}^{\infty}=\left\{e_k^{\perp}\right\}_{k=1}^{\infty}\cup(\{e_k\}_{k=1}^{\infty}\setminus\{x_k\}_{k=1}^{\infty})$, and observe that $\{f_n\}_{n=1}^{\infty}=\{x_k\}_{k=1}^{\infty}\cup\left\{x_k^{\perp}\right\}_{k=1}^{\infty}$.

Now construct a candidate for the desired unitary operator in the theorem. Define a unitary operator $U_\infty:\mathcal{H}\to L_2$ on the basis vectors by setting

$$U_\infty x_k^{\perp}=h_k,\quad U_\infty x_k=g_k\quad\text{for all } k\in\mathbb{N}, \tag{16}$$

in the harmless assumption that, for each $k\in\mathbb{N}$,

$$U_\infty f_k=u_k,\quad U_\infty e_k^{\perp}=h_{n(k)}, \tag{17}$$

where $\{n(k)\}$ is just that sequence which occurs in Condition (c).

Step 2. The verification that U_∞ in (16) has the desired properties is straightforward. Fix an arbitrary $r\in\mathbb{N}$ and put $T=U_\infty S_r U_\infty^{-1}$. Once this is done, the index r may be omitted for S_r, J_r, Q_r, Γ_r.

Write the Schmidt decomposition

$$J=\sum_n s_n\left\langle\cdot,p_n\right\rangle_{\mathcal{H}}q_n,$$

where the s_n are the singular values of J (eigenvalues of $(JJ^*)^{\frac12}$), $\{p_n\}$, $\{q_n\}$ are orthonormal sets (the p_n are eigenvectors for J^*J and the q_n are eigenvectors for JJ^*).

Introduce an auxiliary operator A by

$$A=\sum_n s_n^{\frac14}\left\langle\cdot,p_n\right\rangle_{\mathcal{H}}q_n, \tag{18}$$

and observe that, by the Schwarz inequality,

$$\begin{aligned}\|Af\|_{\mathcal{H}}&=\left\|(J^*J)^{\frac18}f\right\|_{\mathcal{H}}\le\|Jf\|_{\mathcal{H}}^{\frac14},\\ \|A^*f\|_{\mathcal{H}}&=\left\|(JJ^*)^{\frac18}f\right\|_{\mathcal{H}}\le\|J^*f\|_{\mathcal{H}}^{\frac14}\end{aligned} \tag{19}$$

if $\|f\|=1$.

Since $\{e_k^{\perp}\}_{k=1}^{\infty}$ is an orthonormal basis for $(1-E)H$, (3) implies that

$$Q=\sum_k\left\langle\cdot,S^*e_k^{\perp}\right\rangle_{\mathcal{H}}e_k^{\perp}.$$

Whence, using (17), one can write

$$Pf=\sum_k\left\langle f,T^*h_{n(k)}\right\rangle h_{n(k)}\quad(f\in L_2) \tag{20}$$

where $P=U_\infty QU_\infty^{-1}$. By (6),

$$T^*h_{n(k)}=\sum_n\left\langle S^*e_k^{\perp},f_n\right\rangle_{\mathcal{H}}u_n=k\sum_n\left\langle e_k^{\perp},\Gamma f_n\right\rangle_{\mathcal{H}}u_n\quad(k\in\mathbb{N}). \tag{21}$$

Prove that, for any fixed i, the series

$$\sum_n\left\langle e_k^{\perp},\Gamma f_n\right\rangle_{\mathcal{H}}[u_n]^{(i)}(s)\quad(k\in\mathbb{N})$$

converge in the norm of $C(\mathbb{R},\mathbb{C})$. Indeed, all these series are pointwise dominated on $\mathbb{R}$ by one series

$$\sum_n \|\mathit{\Gamma} f_n\|_{\mathcal{H}} \left|[u_n]^{(i)}(s)\right|,$$

which converges uniformly in $\mathbb{R}$ because its component subseries

$$\sum_k \|\mathit{\Gamma} x_k\|_{\mathcal{H}} \left|[g_k]^{(i)}(s)\right|, \quad \sum_k \left\|\mathit{\Gamma} x_k^{\perp}\right\|_{\mathcal{H}} \left|[h_k]^{(i)}(s)\right|$$

are in turn dominated by the convergent series

$$\sum_k d(x_k) G_{k,i}, \quad \sum_k \|\mathit{\Gamma}\| H_{k,i},$$

respectively (see (16), (5), (11), (10)). Whence it follows via (21) that, for each $k \in \mathbb{N}$,

$$\left\|\left[T^* h_{n(k)}\right]^{(i)}\right\|_{C(\mathbb{R},\mathbb{C})} \leq C_i k, \tag{22}$$

with a constant C_i independent of k. Consider functions $\boldsymbol{P} : \mathbb{R}^2 \to \mathbb{C}$, $\boldsymbol{p} : \mathbb{R} \to L_2$, defined, for all $s, t \in \mathbb{R}$, by

$$\begin{aligned} \boldsymbol{P}(s,t) &= \sum_k \left[h_{n(k)}\right](s)\overline{\left[T^* h_{n(k)}\right](t)}, \\ \boldsymbol{p}(s) &= \overline{\boldsymbol{P}(s,\cdot)} = \sum_k \overline{\left[h_{n(k)}\right](s)} T^* h_{n(k)}. \end{aligned} \tag{23}$$

The termwise differentiation theorem implies that, for each i and each integer $j \in [0, +\infty)$,

$$\begin{aligned} \frac{\partial^{i+j}\boldsymbol{P}}{\partial s^i \partial t^j}(s,t) &= \sum_k \left[h_{n(k)}\right]^{(i)}(s)\overline{\left[T^* h_{n(k)}\right]^{(j)}(t)}, \\ \frac{d^i \boldsymbol{p}}{ds^i}(s) &= \sum_k \overline{\left[h_{n(k)}\right]^{(i)}(s)} T^* h_{n(k)}, \end{aligned}$$

since, by (22) and (12), the series displayed converge (absolutely) in $C(\mathbb{R}^2,\mathbb{C})$, $C(\mathbb{R},L_2)$, respectively. Thus, $\dfrac{\partial^{i+j}\boldsymbol{P}}{\partial s^i \partial t^j} \in C(\mathbb{R}^2,\mathbb{C})$, and $\dfrac{d^i\boldsymbol{p}}{ds^i} \in C(\mathbb{R},L_2)$. Observe also that, by (12) and (23), the series (20) (viewed, of course, as one with terms belonging to $C(\mathbb{R},\mathbb{C})$) converges (absolutely) in $C(\mathbb{R},\mathbb{C})$-norm to the function

$$[Pf](s) \equiv \langle f, \boldsymbol{p}(s)\rangle \equiv \int_{\mathbb{R}} \boldsymbol{P}(s,t) f(t)\, dt.$$

Thus, P is an integral operator, and $\boldsymbol{P}$ is its SK^∞-kernel.

Since $\|S^* e_k\|_{\mathcal{H}} = \|J e_k\|_{\mathcal{H}}$ for all k (see (3)), from (2) it follows via (19) that the operator A defined in (18) is nuclear, and hence

$$\sum_n s_n^{\frac{1}{2}} < \infty. \tag{24}$$

Then, according to (18), a kernel which induces the nuclear operator $F = U_\infty J^* U_\infty^{-1}$ can be represented by the series

$$\sum_n s_n^{\frac{1}{2}} U_\infty A^* q_n(s) \overline{U_\infty A p_n(t)} \tag{25}$$

convergent almost everywhere in $\mathbb{R}^2$. The functions used in this bilinear expansion can be written as the series convergent in L_2:

$$U_\infty A p_k = \sum_n \langle p_k, A^* f_n \rangle_{\mathcal{H}} u_n, \quad U_\infty A^* q_k = \sum_n \langle q_k, A f_n \rangle_{\mathcal{H}} u_n \quad (k \in \mathbb{N}).$$

Show that, for any fixed i, the functions $[U_\infty A p_k]^{(i)}$, $[U_\infty A^* q_k]^{(i)}$ ($k \in \mathbb{N}$) make sense, are all in $C(\mathbb{R}, \mathbb{C})$, and their $C(\mathbb{R}, \mathbb{C})$-norms are bounded independent of k. Indeed, all the series

$$\sum_n \langle p_k, A^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, A f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})$$

are dominated by one series

$$\sum_n (\|A^* f_n\|_{\mathcal{H}} + \|A f_n\|_{\mathcal{H}}) \left| [u_n]^{(i)}(s) \right|.$$

This series converges uniformly in $\mathbb{R}$, since it consists of two uniformly convergent in $\mathbb{R}$ subseries

$$\sum_k (\|A^* x_k\|_{\mathcal{H}} + \|A x_k\|_{\mathcal{H}}) \left| [g_k]^{(i)}(s) \right|,$$
$$\sum_k \left(\left\|A^* x_k^\perp\right\|_{\mathcal{H}} + \left\|A x_k^\perp\right\|_{\mathcal{H}} \right) \left| [h_k]^{(i)}(s) \right|,$$

which are dominated by the following convergent series

$$\sum_k d(x_k) G_{k,i}, \quad \sum_k 2\|A\| H_{k,i},$$

respectively (see (5), (19), (11), (10)). Thus, for functions $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{C}$, $\boldsymbol{f} : \mathbb{R} \to L_2$, defined by

$$\boldsymbol{F}(s,t) = \sum_n s_n^{\frac{1}{2}} [U_\infty A^* q_n](s) \overline{[U_\infty A p_n](t)},$$
$$\boldsymbol{f}(s) = \overline{\boldsymbol{F}(s,\cdot)} = \sum_n s_n^{\frac{1}{2}} \overline{[U_\infty A^* q_n](s)} U_\infty A p_n,$$

one can write, for all non-negative integers i, j and all s, $t \in \mathbb{R}$,

$$\frac{\partial^{i+j} \boldsymbol{F}}{\partial s^i \partial t^j}(s,t) = \sum_n s_n^{\frac{1}{2}} [U_\infty A^* q_n]^{(i)}(s) \overline{[U_\infty A p_n]^{(j)}(t)},$$
$$\frac{d^i \boldsymbol{f}}{ds^i}(s) = \sum_n s_n^{\frac{1}{2}} \overline{[U_\infty A^* q_n]^{(i)}(s)} U_\infty A p_n,$$

where the series converge in $C(\mathbb{R}^2, \mathbb{C})$, $C(\mathbb{R}, L_2)$, respectively, because of (24). This implies that $\boldsymbol{F}$ is a SK^∞-kernel of F.

In accordance with (4), we have, for each $f \in L_2$,

$$(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{P}(s,t)f(t)\,dt + \int_{\mathbb{R}} \boldsymbol{F}(s,t)f(t)\,dt$$
$$= \int_{\mathbb{R}} (\boldsymbol{P}(s,t) + \boldsymbol{F}(s,t))f(t)\,dt$$

for almost every s in $\mathbb{R}$. Therefore T is a Carleman operator, and that kernel $\boldsymbol{K}$ of T, which is defined by $\boldsymbol{K}(s,t) = \boldsymbol{P}(s,t) + \boldsymbol{F}(s,t)$ ($s,\ t \in \mathbb{R}$), inherits the SK^∞-kernel properties from its terms. Consequently, $\boldsymbol{K}$ is a SK^∞-kernel of T.

Since scalar factors do not alter the relevant smoothness conditions, the Carleman operators $U_\infty B_r U_\infty^{-1} = rU_\infty S_r U_\infty^{-1}$ ($r \in \mathbb{N}$) have SK^∞-kernels as well. The proof of the theorem is complete.

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